A FFT

Getting the normalizations right is always hard, so let's try to write this down so we can make sure we agree, and also so that 2013's Andreu don't have to do all this again.

A.1 Continous field

We have a continous real field $g(\mathbf{r})$, and we can define its Fourier transform

$$G(\mathbf{k}) \equiv \int d\mathbf{r}\, e^{i\mathbf{kr}}\, g(\mathbf{r}) \ . \tag{A.1}$$

We can also write the inverse relation:

$$g(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\mathbf{k}\, e^{-i\mathbf{kr}}\, G(\mathbf{k}) \ . \tag{A.2}$$

We can also define its correlation function

$$\langle g(\mathbf{r})\, g(\mathbf{r}') \rangle = \xi(|\mathbf{r} - \mathbf{r}'|) \tag{A.3}$$

and its power spectrum

$$\langle G(\mathbf{k})\, G^*(\mathbf{k}') \rangle = (2\pi))^3\ \delta^D(\mathbf{k} - \mathbf{k}')\, P(\mathbf{k}) \ . \tag{A.4}$$

Both are related

$$P(\mathbf{k}) \equiv \int d\mathbf{r}\, e^{i\mathbf{kr}}\, \xi(\mathbf{r}) \ . \tag{A.5}$$

A.2 Discrete field

Now we want to describe the field in a box of volume L^3 and N^3 cells, each of volume $\Delta V = (\Delta r)^3 = (L/N)^3$.

The position of the center of the cell (a, b, c) will be

$$\mathbf{r}_{abc} = (a\Delta r, b\Delta r, c\Delta r) \qquad a, b, c = 0, N-1 \ . \tag{A.6}$$

In each cell we will define the value of the field averaged in its volume

$$f_{abc} = \int d\mathbf{r}\, w_{abc}(\mathbf{r})\, g(\mathbf{r}) \sim g(\mathbf{r}_{abc}) \tag{A.7}$$

where $w_{abc}(\mathbf{r})$ is the normalized selection function, equal to ΔV^{-1} inside the cell and zero outside, and we have made the approximation assuming that the field is quite smooth inside the cell.

We can now describe the FFT of the discrete field:

$$F_{lmn} \equiv \sum_{abc} e^{i\mathbf{k}_{lmn}\mathbf{r}_{abc}}\, f_{abc} \tag{A.8}$$

where we have also defined the wavelength number vector

$$\mathbf{k}_{lmn} = (l\Delta k, m\Delta k, n\Delta k) \ , \qquad l, m, n = 0, N-1 \tag{A.9}$$

and $\Delta k = 2\pi/L$. Note the relation

$$\mathbf{k}_{lmn}\mathbf{r}_{abc} = \frac{2\pi(la + mb + cn)}{N} \tag{A.10}$$

implying that adding or subtraction N to any of the indices of F_{lmn} would not change its value.

Also, since f_{abc} is real, its FFT has to satisfy the relation [1]

$$F^*_{l,m,n} = F_{N-l,N-m,N-n} = F_{-l,-m,-n} \ . \tag{A.11}$$

We can recover the field in configuration space by doing an inverse FFT:

$$f_{abc} = \frac{1}{N^3} \sum_{lmn} e^{-i\mathbf{k}_{lmn}\mathbf{r}_{abc}} \, F_{lmn} \ . \tag{A.12}$$

How does F_{lmn} relates to $G(\mathbf{k}_{lmn})$? Again, in the limit that the field is smoothed at a scale smaller than the cell sice we have

$$G(\mathbf{k}_{lmn}) = \int d\mathbf{r} \, e^{i\mathbf{k}_{lmn}\mathbf{r}} \, g(\mathbf{r}) \sim \Delta V \sum_{abc} e^{i\mathbf{k}_{lmn}\mathbf{r}_{abc}} \, f_{abc} = \Delta V \, F_{lmn} \ . \tag{A.13}$$

Note that $G(\mathbf{k})$ has dimension of $[L^3]$, while F_{lmn} is dimensionless.

From now on we will identify the triplets (a, b, c) by r and the triplets (l, m, n) by k. We can now compute the covariance of the discrete Fourier modes:

$$\begin{aligned}
\langle F_k \, F^*_{k'} \rangle =& \sum_r \sum_{r'} e^{i\mathbf{k}_k\mathbf{r}_r} \, e^{-i\mathbf{k}_{k'}\mathbf{r}_{r'}} \, \langle f_r \, f_{r'} \rangle \\
=& \sum_r \sum_{r''} e^{i\mathbf{k}_k\mathbf{r}_r} \, e^{-i\mathbf{k}_{k'}(\mathbf{r}_{r''}+\mathbf{r}_r)} \xi(\mathbf{r}_{r''}) \\
=& \sum_{r''} e^{-i\mathbf{k}_{k'}\mathbf{r}_{r''}} \xi_{r''} \sum_r e^{i(\mathbf{k}_k-\mathbf{k}_{k'})\mathbf{r}_r} \\
=& N^3 \, \delta^K_{kk'} \, P_k \ ,
\end{aligned} \tag{A.14}$$

where we have $\delta^K_{kk^{prime}}$ is the 3D Kronecker delta function that is one if the three indices are equal and zero otherwise, and that can be expressed as

$$\delta^K_{kk'} = \frac{1}{N^3} \sum_r e^{i(\mathbf{k}_k-\mathbf{k}_{k'})\mathbf{r}_r} \ . \tag{A.15}$$

Finally, we have also defined the FFT of the correlation function

$$P_k \equiv \sum_r e^{i\mathbf{k}_k\mathbf{r}_r} \xi_r \sim \frac{1}{\Delta V} P(\mathbf{k}_k) \ . \tag{A.16}$$

Note that we can recover equation A.4 from A.14 by using the relation between the Kronecker and the Dirac delta functions

$$(2\pi)^3 \ \delta^D(\mathbf{k}) \equiv \int d\mathbf{r} \, e^{i\mathbf{k}\mathbf{r}} \sim \Delta V \sum_r e^{i\mathbf{k}\mathbf{r}_r} \equiv L^3 \, \delta^K_k \ . \tag{A.17}$$

The average amplitude of a given discrete Fourier mode can be computed from the continous power spectrum:

$$\langle F_k \, F^*_k \rangle = N^3 \, P_k = \frac{N^6}{L^3} P(\mathbf{k}) \ . \tag{A.18}$$

[1] We will use commas between the indices only when required to avoid confusion

A.3 Coding

[PM: *(from an e-mail) This $L/N/N$ thing only becomes relevant when you actually want to transform the field - viewed as a simple field in Fourier space, this scaling with N doesn't make any sense (e.g., suppose someone told you you couldn't measure mass density in a cell in g/cm^3, you had to measure it in $g/\mathrm{cm}^3/N$, so the actual number for the same density changed when you changed the size of the box...). Anyway, of course at some point you convert to something your FT software thinks is normalized correctly to be the FT of a real space field if you're going to FT it, but for, e.g., testing power estimation code, this is what you want (i.e., otherwise you would just need to immediately take that factor back out).*]

So in the code now we generate modes that have an amplitude

$$\langle H_k\, H_k^* \rangle = P(\mathbf{k}_k) = \frac{L^3}{N^3} P_k \ , \tag{A.19}$$

i.e., they are related to the previous modes F_k by

$$H_k = \frac{L^{3/2}}{N^3} F_k \ , \tag{A.20}$$

and to compute the field in configuration space we have to do

$$f_r = \frac{1}{N^3} \sum_k e^{-i\mathbf{k}_k \mathbf{r}_r}\, F_k = \frac{1}{L^{3/2}} \sum_k e^{-i\mathbf{k}_k \mathbf{r}_r}\, H_k \ . \tag{A.21}$$

A.4 Test Variance

The variance in the continous field is defined as the correlation function at zero separation,

$$\sigma_g^2 = \left\langle g(\mathbf{r})^2 \right\rangle = \frac{1}{(2\pi)^3} \int d\mathbf{k}\, P(\mathbf{k}) \ . \tag{A.22}$$

In the discrete field, we can compute the variance of a given box,

$$\begin{aligned}
\sigma_f^2 &= \frac{1}{N^3} \sum_r f_r^2 \\
&= \frac{1}{N^3} \sum_r \frac{1}{N^3} \sum_k e^{i\mathbf{k}_k \mathbf{r}_r}\, F_k \frac{1}{N^3} \sum_{k'} e^{i\mathbf{k}_{k'} \mathbf{r}_r}\, F_{k'} \\
&= \frac{1}{N^6} \sum_k F_k\, F_k^* \\
&= \frac{1}{L^3} \sum_k H_k\, H_k^*
\end{aligned} \tag{A.23}$$

B Velocity field

In linear theory, the velocity field is closely related to the density field:

$$\nabla \mathbf{v}(\mathbf{r}) = i\, C\, \delta(\mathbf{r}) \ , \tag{B.1}$$

where C is a proportional constant that depends on the cosmological model and the redshift (LinearUniverse will take care of this).

In Fourier space ∇ is substituted by $-i\mathbf{k}$ and we get

$$\mathbf{v}(\mathbf{k}) = i\,C\,\frac{\mathbf{k}}{k^2}\,\delta(\mathbf{k}) \ , \tag{B.2}$$

Since the velocity field is also real, its Fourier transform will also satisfy the condition

$$\mathbf{v}(-\mathbf{k}) = -i\,C\,\frac{\mathbf{k}}{k^2}\,\delta(-\mathbf{k}) = \mathbf{v}^*(\mathbf{k}) \ . \tag{B.3}$$

B.1 Discrete velocity field

For each density Fourier mode $F_{lmn} \sim F(\mathbf{k}_{lmn})$ we can compute a Fourier mode for each of the components of the velocity, for instance for the x direction we will have:

$$V_{lmn} = i\,C\,\frac{k_l}{|\mathbf{k}_{lmn}|^2} F_{lmn} \ . \tag{B.4}$$

There are several density modes that are purely real, for instance those with $l, m, n = 0$ or $N/2$. The equation above implies that for these wave numbers the velocity modes will be purely imaginary, breaking the relation above!

The mode $\mathbf{k} = 0$ is fine since the velocity modes are not well defined for $\mathbf{k} = 0$, but what about the other modes?

Asking Romain Thessier in the corridor he answered that these are also 0 but I still don't see why...